IS THE SCATTERING AMPLITUDE ANALYTIC IN A FIELD THEORY WITH A COMPACT SPATIAL COORDINATE?

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MOTIVATION

- **INTRODUCTION**
- **POTENTIAL SCATTERING WITH A COMPACT COORDINATE: KHURI'S RESULT**
- **A SCALAR FIELD IN $D = 5$ DIMENSIONS.**
- **COMPACTIFICATION OF $D = 5$ THEORY TO $R^{3,1} \otimes S^1$**
- **ANALYTICITY PROPERTIES OF ELASTIC SCATTERING AMPLITUDE**
- **SUMMARY AND CONCLUSIONS**
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- Let us recapitulate the known rigorous results for $D = 4$ theories derived from axiomatic field theory.
  The Froissart bound
  $$\sigma_t \leq \frac{4\pi}{t_0} ln^2 \frac{s}{s_0}$$
  t_0 is a parameter derived from first principle ($t_0 = 4m_\pi^2$) for most hadronic processes. s_0 is energy scale to make argument of *log* dimensionless and cannot be determined from axiomatic field theoretic frame work. The bound is arrived at from the following ingredients which can be derived from axiomatic field theory.

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- 1. Analyticity of scattering amplitude, $F(s,t)$, in the cut s-plane. $|F(s,t)| \leq s^N, N \in Z$, it is polynomially bounded (EGM) and it satisfies dispersion relation for t inside Lehmann-Martin ellipse.
  2. Crossing symmetry.
  3. Convergence of partial wave amplitude inside Lehmann-Martin ellipse.
  4. Unitarity. The partial wave amplitudes satisfy positivity condition

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- In order to derive statements (1) to (4) above one adopts following steps in the frameworks of general field theories, say LSZ.
- The scattering amplitude, $F(s,t)$, is the boundary value of an analytic function such that

  $$F(s,t) = lim_{\epsilon \to 0} F(s + i\epsilon, t)$$

  with a right hand cut starting from the threshold, s_{thr}, (say $4m^2$) and a left hand cut starting from $u = u_{thr}$. Partial wave expansion:

  $$F(s,t) = \frac{k}{\sqrt{s}} \sum_{l=0}^{\infty} (2l+1) f_l(s) Pl(cos\theta)$$

  $P_l(cos\theta)$ converges for $-1 \leq cos\theta \leq +1$; $t = -2k^2(1 - cos\theta)$ and $\theta =$c.m scattering angle. However, if we intend to study analyticity property of $F(s,t)$ in s for fixed t; there is a problem: as $s \to s_{thr}$, $cos\theta$ goes out of allowed region.

- The resolution: Lehmann proved that the absorptive part of the scattering amplitude has a bigger domain of convergence in the $cos\theta$ plane - the Lehmann ellipse. Then a fixed t dispersion relation can be written for $t \in LE$. More: there is a domain in t-plane with $|t| < \bar{R}$, $\bar{R}$ is s-independent (Martin) such that partial wave expansion converges in an ellipse whose focii are at ± 1 and the semimajor axis is $1 + \frac{t_0}{2k^2}$, k is c.m. momentum. This leads to proof of Froissart bound as we know today (Martin). .
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- There is a lot of interest in theories which live in $D > 4$. However, the extra spatial dimensions must be compactified to get effective 4-dimensional theories for describe experimental data available now.

Khuri's Results

- Khuri [Ann. Phys. **242**, 471(1995)] studied scattering in a quantum mechanical model where a spatial dimension is compactified on a circle, S^1, of radius R; $\frac{1}{R} << 1$. He derived expression for the scattering amplitude in the frame work of perturbation theory. I shall outline his approach and then state his conclusion.
  The spatial geometry is $R^3 \otimes S^1$. The potential is $V(r, \Phi)$. $\mathbf{r} \in \mathbf{R^3}$, $r = |\mathbf{r}|$ and Φ has period 2π. $V(r, \Phi)$ is such that as $r \to \infty$ $V(r, \Phi) \to 0$. In the perturbative frame work he shows that forward scattering amplitude violates analyticity properties for a class of potentials in certain situations. However, a model without S^1 compactification, with same potential (in $d = 3$) has good analyticity properties known from 1957.

- The scattering amplitude depends on three variables - the momentum of the particle, k, the scattering angle θ, and an integer n which appears due to the periodicity of the Φ-coordinate. Thus forward scattering amplitude is denoted by $T_{nn}(K)$, where $K^2 = k^2 + \frac{n^2}{R^2}$. The starting point is the Schrödinger equation

$$\left[\nabla^2 + \frac{1}{R^2}\frac{\partial^2}{\partial \boldsymbol{\Phi}^2} + K^2 - V(r, \Phi)\right]\Psi(\mathbf{r}, \Phi) = 0$$

The free plane wave solutions are

$$\Psi_0(\mathbf{x}, \Phi) = \frac{1}{(2\pi)^2} e^{i\mathbf{k}.\mathbf{x}} e^{in\Phi}$$

and $n \in \mathbf{Z}$. The total energy is

$$\mathbf{K}^2 = k^2 + \frac{n^2}{R^2}$$

- The free Green's function in this case has the following form

$$G_0(\mathbf{K};\mathbf{x},\Phi:\mathbf{x}',\Phi') = -\frac{1}{(2\pi)^4}\sum_{n=-\infty}^{n=+\infty}\int d^3p\,\frac{e^{i\mathbf{p}.(\mathbf{x}-\mathbf{x}')}e^{in(\Phi-\Phi')}}{[p^2+\frac{n^2}{R^2}-\mathbf{K}^2-i\epsilon]}$$

It satisfies the free Schrödinger equation

$$G_0(\mathbf{K};\mathbf{x}-\mathbf{x}';\Phi-\Phi') = -\frac{1}{(8\pi^2)}\sum_{n=-\infty}^{n=+\infty}\frac{e^{i\sqrt{K^2-(n^2/R^2)}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}e^{in(\Phi-\Phi')}$$

Khuri's prescription: $\sqrt{K^2-n^2/R^2}$ is defined in such a way that when $n^2/R^2 > K^2$

$$i\sqrt{K^2-n^2/R^2}\rightarrow -\sqrt{n^2/R^2-K^2},\quad n^2 > K^2R^2$$

Expansion for $G_0(\mathbf{K};\mathbf{x}-\mathbf{x}';\Phi-\Phi')$ is damped for large enough $|n|$. The Green's function, G_0, satisfies the properties satisfied by those in usual potential scattering for fixed k^2.

- The scattering integral equation:

$$\Psi_{k,n}(\mathbf{x},\Phi) = e^{i\mathbf{k}.\mathbf{x}} e^{in\Phi} + \int_0^{2\pi} d\Phi' \int d^3\mathbf{x}' G_0(\mathbf{K}; |\mathbf{x}-\mathbf{x}'|; |\Phi-\Phi'|) \times V(\mathbf{x}',\Phi')\Psi_{k,n}(\mathbf{x}',\Phi')$$

Then we extract the expression for scattering amplitude from large $|\mathbf{x}|$ limit and look at the asymptotic behavior of the wave function,

$$\Psi_{\mathbf{k},n} \to e^{i\mathbf{k}.\mathbf{x}} e^{in\Phi} + \sum_{m=-[KR]}^{+[KR]} T(\mathbf{k}', m : \mathbf{k}, n) \frac{e^{ik'_{mn}|\mathbf{x}|}}{|\mathbf{x}|} e^{im\Phi}$$

$[KR]$: the largest integer less than KR with

$$k'_{mn} = \sqrt{k^2 + \frac{n^2}{R^2} - \frac{m^2}{R^2}}$$

- Therefore, $K^2 = k^2 + (n^2/R^2) = k'^2 + (m^2/R^2)$.
  Remark: The scattered wave has only $(2[KR] + 1)$ components and those states with $m^2/R^2 > k^2 + (n^2/R^2)$ are exponentially damped for large $|\mathbf{x}|$ and consequently these do not appear in the scattered wave. Now we can extract the scattering amplitude to be

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$$T(\mathbf{k}', n'; \mathbf{k}, n) = \quad -\frac{1}{8\pi^2}\int d^3\mathbf{x}' \int_0^{2\pi} d\Phi' e^{-i\mathbf{k}'.\mathbf{x}'} e^{-in'\Phi'} \times V(\mathbf{x}', \Phi')\Psi_{\mathbf{k},n}(\mathbf{x}', \Phi')$$

with the constraint $k'^2 + n'^2/R^2 = k^2 + n^2/R^2$ The reaction is incoming wave $|\mathbf{k}, n>$ is scattered to final state $|\mathbf{k}', n'>$. We shall see how it looks in QFT.

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- Formally, in terms of the full Green's function

$$T(\mathbf{k}', n'; \mathbf{k}, n) - T_B = -\frac{1}{8\pi^2}\int \ldots \int d^3\mathbf{x} d^3\mathbf{x}' d\Phi d\Phi' e^{-i(\mathbf{k}'.\mathbf{x}' + n'\Phi')} V(\mathbf{x}', \Phi') G(\mathbf{K}; \mathbf{x}', \mathbf{x}; \Phi', \Phi) V(\mathbf{x}, \Phi) e^{i(\mathbf{k}.\mathbf{x} + n\Phi)}$$

  T_B is the Born term.

- The full Green's function satisfies equation

$$\left[\nabla^2 + \frac{1}{R^2}\frac{\partial^2}{\partial \boldsymbol{\Phi}^2} + K^2 - V(\mathbf{x}, \Phi)\right] G(\mathbf{K}; \mathbf{x}, \mathbf{x}', \Phi, \Phi') = \delta^3(\mathbf{x} - \mathbf{x}') \delta(\Phi - \Phi')$$

Khuri's conclusion:explicitly computed the second born term $T^{(2)}$ for the forward amplitude, for the choice $n = 1$. With counter examples he showed that the analyticity of this forward amplitude is violated when he chose Yukawa-type potentials of the form

$$V(r, \Phi) = u_0(r) + 2\sum_{m=1}^{N} u_m(r) cos(m\Phi)$$

where $u_m(r) = \lambda_m \frac{e^{-\mu r}}{r}$.
Key Steps: (i) Khuri checked analyticity property of the Green's function and they are analytic. (ii) He studied analyticity of scattering amplitude and found that for $n = 1$, $T(\mathbf{k}, n; \mathbf{k}, n)$, the forward scattering is nonanalytic at the second order. It does not satisfy dispersion relations.

- IMPORTANT POINT: Khuri had studied analyticity property of scattering amplitude for similar Yukawa potentials with no *compact spatial* dimensions [Phys. Rev. **107**, 1148(1957)] also see D. Wang, Phys. Rev. **107**, 350(1957). There was no problem and in fact, they had proved dispersion relation.
  His Remark: If analyticity breaks down (his study is only in potential scattering) and the compactification scale is LARGE can be explored at LHC. Then it will have serious implications for the physics at LHC.

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- **Question:** What is the situation in Relativistic Quantum Field Theory with a Compact Spatial Dimension?

Analyticity Property of Forward Amplitude in a Compactified Field Theory

- How do we proceed?
- Consider a $D = 5$ massive, neutral, scalar field theory in flat space.
- Compactify one spatial dimension on S^1. The geometry is $R^{3,1} \otimes S^1$. Now the spectrum is a massive scalar field of the original theory and tower of KK states.
  Goal: To derive results without appealing to any specific model.
- Assumptions: After KK compactification, there will be tower of states. All particles are stable, there are no bound states, the vacuum is unique for compactified theory

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- We work in the LSZ formulation. The postulates are:

- **A1.** The states of the system are represented in a Hilbert space, $\hat{\mathcal{H}}$. All the physical observables are self-adjoint operators in the Hilbert space, $\hat{\mathcal{H}}$.
  A2. The theory is invariant under inhomogeneous Lorentz transformations.
  A3. The energy-momentum of the states are defined. It follows from the requirements of Lorentz invariance that we can construct a representation of the orthochronous Lorentz group. The representation corresponds to unitary operators, $\hat{U}(\hat{a}, \hat{\Lambda})$, and the theory is invariant under these transformations. Thus there are hermitian operators corresponding to spacetime translations, denoted as $\hat{P}_{\hat{\mu}}$, with $\hat{\mu} = 0, 1, 2, 3, 4$. $[\hat{P}_{\hat{\mu}}, \hat{P}_{\hat{\nu}}] = 0$ If translation operators are chosen to be diagonal we have basis vectors span the Hilbert space

$$\hat{P}_{\hat{\mu}}|\hat{p}, \hat{\alpha} >= \hat{p}_{\hat{\mu}}|\hat{p}, \hat{\alpha} >$$

  Then one has Lorentz invariant vacuum. Another important postulate is microcausality.

- **A4.**. The microcausality: for two bosonic local operators $\mathcal{O}(x)$ *and* $\mathcal{O}(x')$

$$\left[\mathcal{O}(\hat{x}), \mathcal{O}(\hat{x}')\right] = 0, \qquad \textit{for} \;\; (\hat{x} - \hat{x}')^2 < 0$$

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- The asymptotic fields: define $\hat{\phi}(\hat{x})^{in,out}$ which satisfy free field equations. We may construct complete set of states from $\hat{\phi}^{in}$ or $\hat{\phi}^{out}$. $\hat{\phi}(\hat{x})$ is the interacting field; $\hat{\phi}(\hat{x})^{in,out}$ are defined with suitable limiting procedure from $\hat{\phi}(\hat{x})$. The vacuum is unique. Single particle states created by $\hat{\phi}(\hat{x})^{in}$ and $\hat{\phi}(\hat{x})^{out}$ are the same.

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- R-products

$$R\ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n) = \qquad (-1)^n \sum_P \theta(\hat{x}_0 - \hat{x}_{10})...\theta(\hat{x}_{n-10} - \hat{x}_{n0})$$
$$[[...[\hat{\phi}(\hat{x}), \hat{\phi}_{i_1}(\hat{x}_{i_1})], \hat{\phi}_{i_2}(\hat{x}_{i_2})]..], \hat{\phi}_{i_n}(\hat{x}_{i_n})]$$

$R\ \hat{\phi}(\hat{x}) = \hat{\phi}(\hat{x})$; the field is kept where it is. R-product is Lorentz invariant. The VEV of R-product is translationally invariant; consequently, $R(\hat{x}, ..\hat{x}_n)$ depends on difference of coordinates: $\hat{\xi}_1 = \hat{x} - \hat{x}_1$, $\hat{\xi}_2 = \hat{x}_1 - \hat{x}_2$....

- $R^{4,1} \to \; R^{3,1} \otimes S^1$
  Decompose the 5-dimensional spacetime coordinates as:
  $\hat{x}^{\hat{\mu}} = (x^\mu, y), \;\; \mu = 0, 1, 2, 3, \;\; y \in S^1$. Periodicity of y: $y + 2\pi R = y$, R is radius of compactification. Consider, $\hat{\phi}(\hat{x})^{in}$ which satisfies free field equation: $[\Box_5 + m_0^2]\hat{\phi}^{in,out}(\hat{x}) = 0$. We expand the field

$$\hat{\phi}^{in,out}(\hat{x}) = \hat{\phi}^{in,out}(x, y) = \phi_0^{in,out}(x) + \sum_{n=-\infty}^{+n=\infty} \phi_n^{in,out}(x) e^{\frac{iny}{R}}$$

$\phi_0^{in,out}(x)$ has no y-dependence, it is called *zero mode*. For $n \neq 0$

$$[\Box - \frac{\partial}{\partial y^2} + m_n^2]\phi_n^{in,out}(x, y) = 0$$

where $\phi_n^{in,out}(x, y) = \phi_n^{in,out} e^{\frac{iny}{R}}$ and $n = 0$ term is $\phi_0^{in,out}(x) = \phi^{in,out}(x)$ from now on. Here $m_n^2 = m_o^2 + \frac{n^2}{R^2}$.
Momentum associated along y-direction is quantized: $q_n = \frac{n}{R}$; it is additive conserved quantum number.

- Let us look at Källen-Lehmann spectral reprsentation for the 5-dimensional theory

$$< 0|[\hat{\phi}(\hat{x}),\hat{\phi}(\hat{y})]|0 >= \sum_{\hat{\alpha}}\Big(< 0|\hat{\phi}(0)\hat{\alpha} > e^{-i\hat{p}_{\hat{\alpha}}.(\hat{x}-\hat{y})} \times < \hat{\alpha}|\hat{\phi}(0)|0 > -(\hat{x}\leftrightarrow\hat{y})\Big)$$

If we expand $\hat{\phi}(\hat{x})$ in fourier modes as we have done for $\hat{\phi}(x,y)^{in}$ earlier then we arrive at

$$< 0|[\hat{\phi}(x,y),\hat{\phi}(x',y')]|0 >= < 0|[\phi_0(x)+\sum_{-\infty}^{+\infty}\phi_n(x,y),\ \phi_0(x')+ \sum_{-\infty}^{+\infty}\phi_l(x',y')]|0 >$$

The VEV of a commutator of two $\hat{\phi}$ fields in the (KL) representation decompose as sums of several VEV's. Vacuum has $q_n = 0$ thus terms like $< 0|[\phi_n,\phi_{-n}]|0 >$ are admissible

$$< 0|[\phi_0(x),\phi_0(x')]|0 >, \quad < 0|[\phi_n(x),\phi_{-n}(x')]|0 >, ...$$

- The interacting field satisfies equations of motion with a source current, $\hat{j}(\hat{x})$ and it can be expanded as

$$\hat{j}(x,y) = j(x) + \sum_{n=-\infty}^{n=+\infty} J_n(x) e^{iny/R}$$

$\phi(x)$ and $\phi_n(x)$ interpolate to corresponding *in* and *out* fields. $\phi^{in,out}$ and each of the fields $\phi_n^{in,out}(x)$ create their Fock spaces. For example the single particle (say 'in') states are:

$a^{\dagger,in}\mathbf{k})|0>=|\mathbf{k},k_0,in>, k_0>0;\;\; A^{\dagger,in}(\mathbf{p},q_n|0>=|\mathbf{p},p_0;q_n,in>, p_0>0$

Each sector contains a complete set of states is designated with a conserved charge $q_n = \frac{n}{R}$. Thus $<\mathbf{p}',q_n'|\mathbf{p},q_n>=\delta^3(\mathbf{p}'-\mathbf{p})\delta n', n$. Thus $\hat{\mathcal{H}}$ decomposes as

$$\hat{\mathcal{H}} = \sum \oplus \mathcal{H}_n$$

- Definitions and conventions

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- Field and four momenta associated with $n = 0$ charge are respectively denoted as $\phi(x)$ and k. Fields carrying nonzero charges and four momenta are: $\chi(x)$ and p. The elastic scattering between particles (with charge conservation) are of following types:
  (i) $\phi + \phi' \to \phi + \phi'$
  (ii) $\phi + \chi(n) \to \phi' + \chi'(n)$
  (iii)$\chi(m) + \chi(n) \to \chi'(m) + \chi'(n)$. We shall consider scattering of particles with equal charge reaction (iii) without any loss of generality; with this choice (i) and (iii) describe equal mass scattering whereas (ii) is unequal mass scattering.

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- The Mandelstam variables are:

$$s = (\tilde{p}_a + \tilde{p}_b)^2, \quad t = (\tilde{p}_a - \tilde{p}_d)^2, \quad u = (\tilde{p}_a - \tilde{p}_c)^2$$

  $\mathbf{M}_a^2, \quad \mathbf{M}_b^2, \quad \mathbf{M}_c^2, \quad \mathbf{M}_d^2$, are two or more particle states carrying same quantum number as a, b, c, d.
  $(\mathbf{M}_{ab}, \mathbf{M}_{cd}), \quad (\mathbf{M}_{ac}, \mathbf{M}_{bd}), \quad (\mathbf{M}_{ad}, \mathbf{M}_{bc})$ two or more particle states having quantum numbers of $(ab, cd), (ac, bd), (ad, bc)$ respectively.

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- **Scattering of n=0 States**: This is the well known case of scattering of scalar fields (with some difference)
- Utilize LSZ reduction technique

$$< k_d, k_c \; out|k_b, k_a \; in >= \quad 4k_a^0 k_b^0 \delta^3(\mathbf{k}_d - \mathbf{k}_b)\delta^3(\mathbf{k}_a - \mathbf{k}_c) - \frac{i}{(2\pi)^3}\int d^4x \int d^4x' e^{-i(k_a.x - k_c.x')} K_x K_{x'} < k_d \; out|R(x';x)|k_b \; in >$$

with $R(x';x) = -\theta(x_0 - x_0')[\phi_a(x), \phi_c(x')]$. Although it is *identical particle scattering*, we continue to label particles. Here *a and c* are reduced. The KG operators act on $R(x';x)$ in following ways: when they act on the fields $\phi(x)$ and $\phi(x')$ in the commutator unaffecting $\theta(x_0 - x_0')$ we get $-\theta(x_0 - x_0')[j_a(x), j_b(x')]$. Then there are terms like $\delta(x_0 - x_0')$ + terms with *finite number of* derivatives (Symanzik). A δ function with commutators....The derivatives of delta function, when Fourier transformed, give products momenta. But amplitude is *Lorentz invariant*; thus these could be polynomials in s, t, u. Some of the terms vanish from ETC/locality arguments.

- Polynomials in s, t, u do not affect analyticity properties; if they are present we can use subtractions. We use $K_x K_{x'} < k_d \; out|R(x';x)|k_b \; in >=< k_d \; out|R j_c(x') j_a(x))|k_b \; in >$ keeping this fact in mind.

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- SKIP
  The trick: Consider the matrix element $M(x,x') =< \alpha|[A(x), B(x')]|\beta >$ where α and β are states with momenta P_α and P_β respectively.
  Use translation operation and shift by 'a'. Then $M(x,x') = e^{-i(\beta-\alpha).a} M(x+a, x'+a)$.
  Choose $a = -(x+x')/2$. Then $M(x,x')$ takes the form

$$< \alpha|[A(x), B(x')]|\beta >= \quad < \alpha|[A((x-x')/2), B((x'-x)/2)]|\beta > e^{i(\beta-\alpha).(x+x')/2}$$

  Usually, we write product of two operators in this way. After integrations δ-functions appear. We shall encounter this often.

- Define three (generalized) functions (Retarded, Advanced and Causal)

$$F_R(q) = \int_{\infty}^{+\infty} d^4 z e^{iq.z} \theta(z_0) < Q_f | [j_a(z/2), j(-z/2)] | Q_i >$$

$$F_A(q) = -\int_{\infty}^{+\infty} d^4 z e^{iq.z} \theta(-z_0) < Q_f | [j_a(z/2), j_c(-z/2)] | Q_i >$$

and

$$F_C(q) = \int_{-\infty}^{+\infty} d^4 z e^{iq.z} < Q_f | [j_a(z/2), j_b(-z/2)] | Q_i >$$

Q_i and Q_f are states carry four momenta and they are held fixed; at the moment treat them as parameter. Fourier transform of $F_C(q)$, $\tilde{F}_C(z) = 0$, outside the lightcone $z^2 < 0$. Moreover, $F_C(q) = F_R(q) - F_A(q)$

- We open up the commutator to get $j_a(x)j_b(x') - j_b(x')j_a(x)$ then introduce complete set of states $\sum_n |\mathcal{P}_n\tilde{\alpha}_n >< \mathcal{P}_n\tilde{\alpha}_n| = \mathbf{1}$ and $\sum_{n'} |\bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'} >< \bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'}| = \mathbf{1}$. between products of currents. $\{\tilde{\alpha}_n, \tilde{\beta}_{n'}\}$ are states permitted by energy momentum conservation q_n quantum number conservations. They come from $\sum \oplus \mathcal{H}$. Present case: we should have $n = 0$ from sum of q_n's from each state. $F_C(q)$ is

$$\int d^4 z e^{iq.z}\Big[\sum_n \Big(\int d^4\mathcal{P}_n < Q_f|j_a(\frac{z}{2)}|\mathcal{P}_n\tilde{\alpha}_n >< \mathcal{P}_n\tilde{\alpha}_n|j_c(-\frac{z}{2})|Q_i >$$

$$-\sum_{n'} \Big(\int d^4\bar{\mathcal{P}}_{n'} < Q_f|j_c(-\frac{z}{2})|\bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'} >< \bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'}|j_a(\frac{z}{2})|Q_i > \Big)\Big]$$

- Then

$$F_C(q) = \sum_n \Bigg(< Q_f|j_a(0)|\mathcal{P}_n = \frac{(Q_i + Q_f)}{2} - q, \tilde{\alpha}_n > \times$$

$$< \tilde{\alpha}_n, \mathcal{P}_n = \frac{(Q_i + Q_f)}{2} - q|j_c(0)|Q_i > \Bigg) -$$

$$\sum_{n'} \Bigg(< Q_f|j_c(0)|\bar{\mathcal{P}}_{n'} = \frac{(Q_i + Q_f)}{2} + q, \tilde{\beta}_{n'} > \times$$

$$< \tilde{\beta}_{n'}, \bar{\mathcal{P}}_{n'} = \frac{(Q_i + Q_f)}{2} + q|j_a(0)|Q_i > \Bigg)$$

$\mathcal{P}_n = \frac{(Q_i+Q_f)}{2} - q$.... appear since we have used translation operation and carried out an integration leading to an energy momentum conserving δ-function.

- Remark: This is starting point to prove analyticity of the amplitude for fixed t through J-L-D representation, then existence of Lehmann Ellipses and the Crossing Symmetry. Entire KK tower does not contribute to this sum from energy momentum conservation considerations (details later).
  All these can be achieved for the present case. Therefore, **elastic scattering in the $n = 0$ sector satisfies analyticity**. Khuri came to the same conclusion.
  We can repeat the same calculation for scattering of $n = 0$ particle with $n \neq 0$ particle. We reduce fields ϕ_a and ϕ_c (these are $n = 0$ states. The repeat the above-mentioned prescription. Only difference that it is unequal mass scattering. Thus the analyticity of the amplitude can be proved. This case was not addressed by Khuri.

- **Elastic Scattering of** $\mathbf{n(p_a) + n(p_b) \rightarrow n(p_c) + n(p_d)}$

- **Elastic Scattering of $\mathbf{n(p_a) + n(p_b) \to n(p_c) + n(p_d)}$**
- We can proceed the same way as before. The fields are denoted by $\chi_a, \chi_b, \chi_c, \chi_d$ with respective momenta p_a, p_b, p_c, p_d. Following standard prescription

$$< p_d, p_c \; out | p_b, p_a \; in >= \quad < p_d, p_c \; in | p_b, p_a \; in > - \frac{1}{(2\pi)^3} \int d^4x \int d^4x' e^{-i(p_a.x - p_c.x')} < p_d | \theta(x'_0 - x_0) [J_c(x'), J_a(x)] | p_b >$$

$J_a(x)$ is source current for $\chi_a(x)$ and similarly for $J_c(x')$. Invoke unitarity,

$$F(s,t) = i \int d^4x e^{i(p_a+p_c).\frac{x}{2}} \theta(x_0) < p_d | [J_a(x/2), J_c(-x'/2)] | p_b >$$

We evaluate the imaginary part of this amplitude, $F(s,t,)$

-

$$Im\ F(s,t) = \frac{1}{2i}(F - F^*)$$
$$= \frac{1}{2}\int d^4x e^{i(p_a+p_c).\frac{x}{2}} < p_d|[J_a(x/2), J_c(-x/2)]|p_b >$$

We use the fact that F^* is invariant under interchange $p_b \to p_d$ and also $p_d \to p_b$; $\theta(x_0) + \theta(-x_0) = 1$. Open up the commutator of the two currents; introduce a complete set of states $\sum_{\mathcal{N}} |\mathcal{N} >< \mathcal{N}| = 1$. implement translation operations in each of the (expanded) matrix elements to express arguments of each current as $J_a(0)$ and $J_c(0)$ finally integrate over d^4x to get the δ-functions. Then

$$F(p_d, p_c; p_b, p_a) - F^*(p_b, p_a; p_c, p_d) = 2\pi i \sum_{\mathcal{N}} \Big[\delta(p_d + p_c - p_n) F(p_d, p_c; n) F^*(p_a, p_b; n) - \delta(p_a - p_c - p_n) F(p_d, -p_a; n) F^*(p_b, -p_c; n)\Big]$$

Generalized unitarity relation. Forward case: implies optical theorem.

- Look at the first term: δ function implies $p_a + p_b = p_n = p_c + p_d$. This is s-channel process, $p_n^2 = \mathcal{M}_n^2 = s$.
Look at second term: $p_b + (-p_c) = p_n = p_d + (-pa)$:
$p_n^2 = \mathcal{M}_n^2 = (p_b - p_c)^2$. Go to a Lorentz frame $p_b = (m_b, \mathbf{0})$, then

$$\mathcal{M}_n^2 = 2m_b(m_b - p_c^0), \quad p_c^0 > 0$$

Note: $m_a = m_c$, $p_c^0 = \sqrt{m_c^2 + \mathbf{p}_c^2}$; $\mathcal{M}_n^2 < 0$. $\mathcal{M}_n$ is intermediate physical state carrying n charge. Thus above condition cannot be satisfied. The 2nd term does not contribute to s-channel process. Instead look at cross channel process:

$$p_b + (-p_c) \to p_d + (-p_a); \quad -p_a^0 > 0, \; and \; -p_c^0 > 0$$

p_b and p_c are incoming (hence the negative sign for p_c) and p_d and p_a are outgoing. However, the first term does not contribute. Here is hint of *crossing symmetry* (it is not a proof - can be proved ?). We are not interested to prove crossing symmetry here! The δ-functions guarantee energy momentum conservation. Generalized Unitarity implies there is a cut off for KK towers *as intermediate states* so long as s is finite, s could be very large.

- *Forward Scattering Amplitude: The Analyticity Property*

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- In this case the process is $p_c = p_a$ and $p_d = p_b$. We have forward scattering of equal mass particles, $m_n^2 = m_0^2 + \frac{n^2}{R^2}$. Starting from

$$F(p_b, p_a; p_b, p_a) = \int d^4x e^{ip_a.x} (\Box_x + m_n^2)^2 < p_b | R\chi_a(x)\chi_a(0) | p_b >$$

We arrive at

$$F(p_b, p_a; p_b, p_a) = \int d^4x e^{ip_a.x} < p_b | RJ_a(x)J_a(0) | p_b >$$

We go to the rest frame of particle 'b': $p_b = (m_b, \mathbf{0})$ and define $\omega = \frac{p_a.p_b}{m_n}$. In this frame (adopted by Symanzik)

$$F(p_b, p_a : p_b, p_a) = i \int_0^\infty \int d^3\mathbf{x} e^{ip_a^0 x^0 - i\sqrt{(p_a^0)^2 - m_n^2}\hat{\mathbf{e}}.\mathbf{x}} \tilde{f}(\mathbf{x}, x_0)$$

$\hat{\mathbf{e}}$ is the unit vector along $\mathbf{p}_a$. We can identify $\tilde{f}(\mathbf{x}, x_0)$; and from microcausality, we conclude $\tilde{f}(\mathbf{x}, x_0) = 0$, unless $x_0 > |\mathbf{x}|$.

- After the angular integration

$$F(p_b, p_a; p_b, p_a) = \int_0^\infty \mathcal{F}(\omega, r) dr$$

with

$$\mathcal{F}(\omega, r) = \quad 4\pi i \frac{sin\sqrt{\omega^2 - m_n^2} r e^{i\omega r}}{\sqrt{\omega^2 - m_n^2} r} \times$$
$$\int_r^\infty dt e^{i\omega(r-t)} < p_b | [J_a(x), J_a(0)] | p_b >$$

Technicalities - SKIP: $\mathcal{F}(\omega, r)$ is analytic function of ω for $Im\ \omega \geq 0$ (upper half plane). (i) No branch point at $\omega = \pm m_n$ since $\frac{sin\sqrt{\omega^2 - m_n^2} r}{r\sqrt{\omega^2 - m_n^2}}$ even in $r\sqrt{(\omega^2 - m_n^2}$.
(ii) For, $\omega < m_n$ problem in behavior of $sin\sqrt{\omega^2 - m_n^2} r$? The presence of $e^{i\omega r}$ takes care. (iii) Assume, F is well behaved in s - no subtractions. To write dispersion relation for F, we have to interchange integration over r and ω. Write a dispersion relation for $\mathcal{F}(\omega, r)$ (assume it vanishes for large ω), then

- $$\mathcal{F}(\omega, r) = \frac{1}{\pi}\int_{\infty}^{+\infty} \frac{\mathrm{Im}\ \mathcal{F}(\omega', r)}{\omega' - \omega - i\epsilon} d\omega'$$

Note that $\mathrm{Im}\ \mathcal{F}$ has the property: $\mathrm{Im}\ \mathcal{F}(r, \omega) = -\mathrm{Im}\ \mathcal{F}(-\omega, r)$. The integral is

$$\mathcal{F}(\omega, r) = \frac{1}{\pi}\int_{0}^{+\infty} \mathrm{Im}\ \mathcal{F}(\omega', r)\left[\frac{1}{\omega' - \omega - i\epsilon} + \frac{1}{\omega' - \omega + i\epsilon}\right] d\omega'$$

Now $\mathrm{Im}\ F$ is expressed as

$$\mathrm{Im}\ F(p_b, p_a; p_b, p_a) = \frac{1}{2}\int d^4x e^{p_a . x} < p_b|[J_a(x), J_a(0)]|p_b > \qquad (1)$$

We can open up the commutator, insert complete set of states, use translation operation and carry out the angular integration to get

$$\mathrm{Im}\ F(p_b, p_a; p_b, p_a) = \quad \frac{1}{2}(2\pi)^4 \sum_n | < p_b|J_a(0)|p_b > |^2 \Big]$$
$$\times [\delta^4(p_b + p_a - p_n) - \delta^4(p_b - p_a + p_n) \Big]$$

- The expression for $\mathcal{F}(\omega, r)$ is

$$\mathrm{Im}\ F(p_b, p_a; p_b, p_a) = \frac{1}{2}\int dr 4\pi r^2 \frac{sin\sqrt{\omega^2 - m_n^2}r}{\sqrt{\omega^2 - m_n^2}r} \times \int_{-\infty}^{+\infty} e^{i\omega t} < p_b|[J_a(x), J_a(0)]|p_b > dt$$

While writing dispersion integral for $F(p_a, p_b, p_c, p_d)$ the issue of interchanging t and ω integral comes up. Symanzik has resolved this in his (1957) paper on *forward dispersion relation for πN scattering*. Here is a simple problem of scattering of equal mass bosons. Thus the dispersion relation written above for forward scattering amplitude holds $F(\omega)$. Moreover, Bogoliubov's approach leads to same conclusion.

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- Thus the forward amplitude satisfies dispersion relation. We have assumed good behavior for large ω. We discuss subtractions later. Conclusion of this section: Analyticity is not violated. This is different from the conclusion of Khuri who studied analyticity of amplitude perturbatively in potential scattering.

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- The elastic scatterings amplitudes for (i) $(n = 0) + (n = 0)$ and (ii) $(n = 0) + (n \neq 0)$ satisfy analyticity properties. In fact all the known results of analyticity in LSZ frame work can be derived (not proved here).

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- The elastic scatterings amplitudes for (i) $(n=0)+(n=0)$ and (ii) $(n=0)+(n \neq 0)$ satisfy analyticity properties. In fact all the known results of analyticity in LSZ frame work can be derived (not proved here).
- We systematically studies forward elastic scattering amplitude of $(n)+(n)$ for $n \neq 0$. We showed that the forward scattering amplitude satisfies dispersion relations.

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- We assumed *no subtractions* - in any case (at most) the amplitude can have polynomial growth and that is fine; we can write N-subtracted dispersion relation.
- How about fixed-t dispersion relation? To go beyond Khuri. In order to prove fixed-t dispersion relation we have derived the analog of the Jost-Lehmann-Dyson representation for the causal commutator of source currents. Next we proved the existence of Lehmann ellipse. Thus the scattering amplitude has an enlarged domain of analyticity beyond $|cos\theta| = 1$. Consequently, analog of the Froissart bound is proved. The Jin-Martin bound is derived, $N < 2$. Be careful about the presence of KK towers while deriving the results.

THANK YOU